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New Explicit Exact Solutions of the Born–Infeld Equation

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By introducing an auxiliary ordinary differential equation and solving it by means of the method of separation of variables, many new explicit exact solutions to the Born–Infeld equation are found in a concise manner.

KEY WORDS: auxiliary ordinary differential equation; Born–Infeld equation; explicit exact solution.

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1. INTRODUCTION

As more and more problems in branches of modern mathematical physics and other interdisciplinary science are described in terms of suitable nonlinear models, directly seeking explicit and exact solutions of nonlinear evolution equations (NEEs for short) plays a very important role in nonlinear science. In recent years, a vast variety of new and powerful approaches have been put forward to construct explicit and exact solutions to NEEs. Among them there are the homogeneous balance method (Wang, 1995, 1996), the tanh-function method (Malfliet, 1992; Parkes and Duffy, 1997; Zhang, 1998), the sech-function method (Duffy and Parkes, 1996), the sine–cosine method (Yan and Zhang, 1999), the trial function method (Liu *et al.*, 2001a; Otwinowski *et al.*, 1988; Xie and Tang, 2004), the Jacobi elliptic function expansion method (Fu *et al.*, 2001; Liu *et al.*, 2001b; Parkes *et al.*, 2002), the mapping method (Peng, 2003) and so on. Unfortunately, not all these methods are universally suitable for solving all kinds of NEEs. As a result, it is still a very significant task to search for more powerful and efficient methods to solve NEEs.

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In the present paper, by using the solutions of an auxiliary ordinary differential equation, we obtain many new explicit exact solutions to the Born–Infeld equation in a simple way.

2. SOLUTIONS TO THE BORN–INFELD EQUATION

The celebrated Born–Infeld equation (Sirendaoreji and Jiong, 2002) reads

$$\left(1-u_t^2\right)u_{xx}+2u_xu_tu_{xt}-\left(1+u_x^2\right)u_{tt}=0 \tag{1}$$

In what follows, we look for its traveling wave solutions of the following form

$$u=u(\xi),\quad \xi=x-\lambda t \tag{2}$$

Substituting Eq. (2) into Eq. (1) leads to the following ordinary differential equation (ODE for short) with respect to ξ

$$[1-\lambda^2(u')^2]u''+2\lambda^2(u')^2u''-\lambda^2[1+(u')^2]u''=0 \tag{3}$$

Here and henceforth, a prime stands for the derivatives for ξ.

In order to solve Eq. (3) easily, here we introduce the following additional ODE which $u(\xi)$ in Eq. (3) satisfies

$$u'=a_0+\sum_{j=1}^{n}(a_j\cos u+b_j\sin u) \tag{4}$$

where $a_j(j=0,1,2,\ldots)$ and $b_j(j=1,2,\ldots)$ are constants to be determined, and the positive integer n can be determined by the leading-order analysis method.

Making use of the following formula

$$\sin u=\frac{e^{iu}-e^{-iu}}{2i}\qquad \cos u=\frac{e^{iu}+e^{-iu}}{2} \tag{5}$$

and inserting Eq. (5) together with Eq. (4) into Eq. (3) and applying the leading-order analysis method, we obtain $n=1$, Eq. (4) can be rewritten as

$$u'=a_0+a_1\cos u+b_1\sin u \tag{6}$$

By means of the method of variable separation and with the help of Mathematica, we may find the following solutions to Eq. (6).

Case 1: $a_1=b_1=0,\qquad a_0\neq 0$

$$u(\xi)=a_0\xi+\xi_0 \tag{7}$$

Case 2: $a_0 = b_1 = 0, \qquad a_1 \neq 0$

$$u(\xi) = \begin{cases} \frac{1}{k}\arcsin[\tanh(ka_1\xi + \xi_0)] \\ \frac{1}{k}\arcsin[\coth(ka_1\xi + \xi_0)] \\ \frac{1}{k}\arcsin[\tanh(2ka_1\xi + \xi_0) \pm i \sec h(2ka_1\xi + \xi_0)] \\ \frac{1}{k}\arcsin[\coth(2ka_1\xi + \xi_0) \pm \csc h(2ka_1\xi + \xi_0)] \\ \frac{1}{k}\arcsin\left[\frac{\cosh(2ka_1\xi + \xi_0) \mp 1}{\sinh(2ka_1\xi + \xi_0)}\right] \\ \frac{1}{k}\arcsin\left[\frac{\sinh(2ka_1\xi + \xi_0)}{\cosh(2ka_1\xi + \xi_0) \pm 1}\right] \\ \frac{1}{k}\arccos[\pm \sec h(ka_1\xi + \xi_0)] \\ \frac{1}{k}\arccos[\pm i \csc h(ka_1\xi + \xi_0)] \end{cases} \tag{8}$$

where $i = \sqrt{-1}$.

Case 3: $a_0 = a_1 = 0, \qquad b_1 \neq 0$

$$u(\xi) = \begin{cases} \frac{1}{k}\arcsin[\pm \sec h(kb_1\xi + \xi_0)] \\ \frac{1}{k}\arcsin[\pm i \csc h(kb_1\xi + \xi_0)] \\ \frac{1}{k}\arccos[\pm \tanh(kb_1\xi + \xi_0)] \\ \frac{1}{k}\arccos[\pm \coth(kb_1\xi + \xi_0)] \\ \frac{1}{k}\arccos[\pm \tanh(2kb_1\xi + \xi_0) \mp i \sec h(2kb_1\xi + \xi_0)] \\ \frac{1}{k}\arccos[\pm \coth(2kb_1\xi + \xi_0) \mp \csc h(2kb_1\xi + \xi_0)] \\ \frac{1}{k}\arccos\left[\pm\frac{\cosh(2kb_1\xi + \xi_0) \mp 1}{\sinh(2kb_1\xi + \xi_0)}\right] \\ \frac{1}{k}\arccos\left[\pm\frac{\sinh(2kb_1\xi + \xi_0)}{\cosh(2kb_1\xi + \xi_0) \pm 1}\right] \end{cases} \tag{9}$$

where $i = \sqrt{-1}$.

Case 4: $a_1 = 0, \qquad a_0 \neq 0, \qquad b_1 \neq 0$

$$u(\xi)=\begin{cases}\frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+b_1}{a_0-b_1}}\tan\frac{k\sqrt{a_0^2-b_1^2}}{2}(\xi+\xi_0)\right]+\frac{\pi}{2k}\\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+b_1}{a_0-b_1}}\cot\frac{k\sqrt{a_0^2-b_1^2}}{2}(\xi+\xi_0)\right]+\frac{\pi}{2k}\\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+b_1}{a_0-b_1}}\Big(\tan k\sqrt{a_0^2-b_1^2}(\xi+\xi_0)\right.\\ \quad\left.\pm\sec k\sqrt{a_0^2-b_1^2}(\xi+\xi_0)\Big)\right]+\frac{\pi}{2k}\\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+b_1}{a_0-b_1}}\Big(\cot k\sqrt{a_0^2-b_1^2}(\xi+\xi_0)\right.\\ \quad\left.\pm\csc k\sqrt{a_0^2-b_1^2}(\xi+\xi_0)\Big)\right]+\frac{\pi}{2k}\\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+b_1}{a_0-b_1}}\frac{1\mp\cos k\sqrt{a_0^2-b_1^2}(\xi+\xi_0)}{\sin k\sqrt{a_0^2-b_1^2}(\xi+\xi_0)}\right]+\frac{\pi}{2k}\\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+b_1}{a_0-b_1}}\frac{\sin k\sqrt{a_0^2-b_1^2}(\xi+\xi_0)}{1\pm\cos k\sqrt{a_0^2-b_1^2}(\xi+\xi_0)}\right]+\frac{\pi}{2k}\end{cases}\quad \text{for } a_0^2>b_1^2 \tag{10}$$

$$u(\xi)=\begin{cases}\frac{2}{k}\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\tanh\frac{k\sqrt{b_1^2-a_0^2}}{2}(\xi+\xi_0)\right]+\frac{\pi}{2k}\\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\coth\frac{k\sqrt{b_1^2-a_0^2}}{2}(\xi+\xi_0)\right]+\frac{\pi}{2k}\\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\Big(\tanh k\sqrt{b_1^2-a_0^2}(\xi+\xi_0)\right.\\ \quad\left.\pm i\sec hk\sqrt{b_1^2-a_0^2}(\xi+\xi_0)\Big)\right]+\frac{\pi}{2k}\\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\Big(\coth k\sqrt{b_1^2-a_0^2}(\xi+\xi_0)\right.\\ \left.\pm\csc hk\sqrt{b_1^2-a_0^2}(\xi+\xi_0)\Big)\right]+\frac{\pi}{2k}\\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\frac{\cosh k\sqrt{b_1^2-a_0^2}(\xi+\xi_0)\mp 1}{\sinh k\sqrt{b_1^2-a_0^2}(\xi+\xi_0)}\right]+\frac{\pi}{2k}\\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\frac{\sinh k\sqrt{b_1^2-a_0^2}(\xi+\xi_0)}{\cosh k\sqrt{b_1^2-a_0^2}(\xi+\xi_0)\pm 1}\right]+\frac{\pi}{2k}\end{cases}\quad \text{for } a_0^2<b_1^2 \tag{11}$$

$$u(\xi)=\frac{2}{k}\arctan ka_0(\xi+\xi_0)+\frac{\pi}{2k}\qquad \text{for } b_1=a_0 \tag{12}$$

$$u(\xi)=-\frac{2}{k}\operatorname{arccot} ka_0(\xi+\xi_0)+\frac{\pi}{2k}\qquad \text{for } b_1=-a_0 \tag{13}$$

Case 5: $b_1 = 0, \quad a_0 \neq 0, \quad a_1 \neq 0$

$$u(\xi) = \begin{cases} \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+a_1}{a_0-a_1}}\tan\frac{k\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right] \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+a_1}{a_0-a_1}}\cot\frac{k\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right] \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+a_1}{a_0-a_1}}\tan k\sqrt{a_0^2-a_1^2}(\xi+\xi_0) \right. \\ \quad \left. \pm\sec k\sqrt{a_0^2-a_1^2}(\xi+\xi_0))\right] \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+a_1}{a_0-a_1}}\left(\cot k\sqrt{a_0^2-a_1^2}(\xi+\xi_0)\right.\right. \\ \quad \left.\left. \pm\csc k\sqrt{a_0^2-a_1^2}(\xi+\xi_0)\right)\right] \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+a_1}{a_0-a_1}}\frac{1\mp\cos k\sqrt{a_0^2-a_1^2}(\xi+\xi_0)}{\sin k\sqrt{a_0^2-a_1^2}(\xi+\xi_0)}\right] \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+a_1}{a_0-a_1}}\frac{\sin k\sqrt{a_0^2-a_1^2}(\xi+\xi_0)}{1\pm\cos k\sqrt{a_0^2-a_1^2}(\xi+\xi_0)}\right] \end{cases} \quad \text{for } a_0^2 > a_1^2 \tag{14}$$

$$u(\xi) = \begin{cases} \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_1+a_0}{a_1-a_0}}\tanh\frac{k\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right] \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_1+a_0}{a_1-a_0}}\coth\frac{k\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right] \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_1+a_0}{a_1-a_0}}\left(\tanh k\sqrt{a_1^2-a_0^2}(\xi+\xi_0)\right.\right. \\ \quad \left.\left. \pm i\sec hk\sqrt{a_1^2-a_0^2}(\xi+\xi_0)\right)\right] \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_1+a_0}{a_1-a_0}}\left(\coth k\sqrt{a_1^2-a_0^2}(\xi+\xi_0)\right.\right. \\ \quad \left.\left. \pm\csc hk\sqrt{a_1^2-a_0^2}(\xi+\xi_0)\right)\right] \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_1+a_0}{a_1-a_0}}\frac{\cosh k\sqrt{a_1^2-a_0^2}(\xi+\xi_0)\pm1}{\sinh k\sqrt{a_1^2-a_0^2}(\xi+\xi_0)}\right] \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_1+a_0}{a_1-a_0}}\frac{\sinh k\sqrt{a_1^2-a_0^2}(\xi+\xi_0)}{\cosh k\sqrt{a_1^2-a_0^2}(\xi+\xi_0)\pm1}\right] \end{cases} \quad \text{for } a_0^2 < a_1^2 \tag{15}$$

$$u(\xi) = \frac{2}{k}\arctan ka_0(\xi+\xi_0) \qquad \text{for} \quad a_1 = a_0 \tag{16}$$

$$u(\xi) = -\frac{2}{k}\operatorname{arccot} ka_0(\xi+\xi_0) \qquad \text{for} \quad a_1 = -a_0 \tag{17}$$

Case 6: $a_0 = 0, \quad a_1 \neq 0, \quad b_1 \neq 0$

$$u(\xi) = \begin{cases} \frac{2}{k}\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}\tanh\frac{k\sqrt{a_1^2+b_1^2}}{2}(\xi+\xi_0)+\frac{b_1}{a_1}\right] \\ \frac{2}{k}\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}\coth\frac{k\sqrt{a_1^2+b_1^2}}{2}(\xi+\xi_0)+\frac{b_1}{a_1}\right] \\ \frac{2}{k}\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}(\tanh k\sqrt{a_1^2+b_1^2}(\xi+\xi_0)\right. \\ \quad \left.\pm i\,\sec hk\sqrt{a_1^2+b_1^2}(\xi+\xi_0))+\frac{b_1}{a_1}\right] \\ \frac{2}{k}\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}(\coth k\sqrt{a_1^2+b_1^2}(\xi+\xi_0)\right. \\ \quad \left.\pm \csc hk\sqrt{a_1^2+b_1^2}(\xi+\xi_0))+\frac{b_1}{a_1}\right] \\ \frac{2}{k}\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}\frac{\cosh k\sqrt{a_1^2+b_1^2}(\xi+\xi_0)\mp 1}{\sinh k\sqrt{a_1^2+b_1^2}(\xi+\xi_0)}+\frac{b_1}{a_1}\right] \\ \frac{2}{k}\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}\frac{\sinh k\sqrt{a_1^2+b_1^2}(\xi+\xi_0)}{\cosh k\sqrt{a_1^2+b_1^2}(\xi+\xi_0)\pm 1}+\frac{b_1}{a_1}\right] \end{cases} \tag{18}$$

Case 7: $a_0 \neq 0, \quad a_1 \neq 0, \quad b_1 \neq 0$

$$u(\xi) = $$

$$\begin{cases} \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+\sqrt{a_1^2+b_1^2}}{a_0-\sqrt{a_1^2+b_1^2}}}\tan\frac{k\sqrt{a_0^2-a_1^2-b_1^2}}{2}(\xi+\xi_0)\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+\sqrt{a_1^2+b_1^2}}{a_0-\sqrt{a_1^2+b_1^2}}}\cot\frac{k\sqrt{a_0^2-a_1^2-b_1^2}}{2}(\xi+\xi_0)\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+\sqrt{a_1^2+b_1^2}}{a_0-\sqrt{a_1^2+b_1^2}}}\Big(\tan k\sqrt{a_0^2-a_1^2-b_1^2}(\xi+\xi_0)\right. \\ \quad \left.\pm\sec k\sqrt{a_0^2-a_1^2-b_1^2}(\xi+\xi_0)\Big)\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+\sqrt{a_1^2+b_1^2}}{a_0-\sqrt{a_1^2+b_1^2}}}\Big(\cot k\sqrt{a_0^2-a_1^2-b_1^2}(\xi+\xi_0)\right. \\ \quad \left.\pm\csc k\sqrt{a_0^2-a_1^2-b_1^2}(\xi+\xi_0)\Big)\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+\sqrt{a_1^2+b_1^2}}{a_0-\sqrt{a_1^2+b_1^2}}}\frac{1\pm\cos k\sqrt{a_0^2-a_1^2-b_1^2}(\xi+\xi_0)}{\sin k\sqrt{a_0^2-a_1^2-b_1^2}(\xi+\xi_0)}\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{a_0+\sqrt{a_1^2+b_1^2}}{a_0-\sqrt{a_1^2+b_1^2}}}\frac{\sin k\sqrt{a_0^2-a_1^2-b_1^2}(\xi+\xi_0)}{1\pm\cos k\sqrt{a_0^2-a_1^2-b_1^2}(\xi+\xi_0)}\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \end{cases} \quad \text{for} \quad a_0^2 > a_1^2+b_1^2 \tag{19}$$

$$u(\xi) = \begin{cases} \frac{2}{k}\arctan\left[\pm\sqrt{\frac{\sqrt{a_1^2+b_1^2}+a_0}{\sqrt{a_1^2+b_1^2}-a_0}}\tanh\frac{k\sqrt{a_1^2+b_1^2-a_0^2}}{2}(\xi+\xi_0)\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{\sqrt{a_1^2+b_1^2}+a_0}{\sqrt{a_1^2+b_1^2}-a_0}}\coth\frac{k\sqrt{a_1^2+b_1^2-a_0^2}}{2}(\xi+\xi_0)\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{\sqrt{a_1^2+b_1^2}+a_0}{\sqrt{a_1^2+b_1^2}-a_0}}(\tanh k\sqrt{a_1^2+b_1^2-a_0^2}(\xi+\xi_0) \right. \\ \quad \left. \pm i\sec hk\sqrt{a_1^2+b_1^2-a_0^2}(\xi+\xi_0)\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \qquad \text{for} \quad a_0^2 < a_1^2+b_1^2 \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{\sqrt{a_1^2+b_1^2}+a_0}{\sqrt{a_1^2+b_1^2}-a_0}}\big(\coth k\sqrt{a_1^2+b_1^2-a_0^2}(\xi+\xi_0) \right. \\ \quad \left. \pm \csc hk\sqrt{a_1^2+b_1^2-a_0^2}(\xi+\xi_0)\big)\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{\sqrt{a_1^2+b_1^2}+a_0}{\sqrt{a_1^2+b_1^2}-a_0}}\frac{\cosh k\sqrt{a_1^2+b_1^2-a_0^2}(\xi+\xi_0)\mp 1}{\sinh k\sqrt{a_1^2+b_1^2-a_0^2}(\xi+\xi_0)}\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \\ \frac{2}{k}\arctan\left[\pm\sqrt{\frac{\sqrt{a_1^2+b_1^2}+a_0}{\sqrt{a_1^2+b_1^2}-a_0}}\frac{\sinh k\sqrt{a_1^2+b_1^2-a_0^2}(\xi+\xi_0)}{\cosh k\sqrt{a_1^2+b_1^2-a_0^2}(\xi+\xi_0)\pm 1}\right]-\frac{1}{k}\arctan\frac{a_1}{b_1}+\frac{\pi}{2k} \end{cases} \tag{20}$$

$$u(\xi) = \frac{2}{k}\arctan ka_0(\xi+\xi_0) - \frac{1}{k}\arctan\frac{a_1}{b_1} + \frac{\pi}{2k} \quad \text{for} \sqrt{a_1^2+b_1^2} = a_0 \tag{21}$$

$$u(\xi) = -\frac{2}{k}\text{arccot}\, ka_0(\xi+\xi_0) - \frac{1}{k}\arctan\frac{a_1}{b_1} + \frac{\pi}{2k} \quad \text{for} \sqrt{a_1^2+b_1^2} = -a_0 \tag{22}$$

Inserting Eq. (6) into Eq. (3) yields

$$\begin{aligned} &a_0a_1(1+\lambda^2)\sin u + a_0b_1(1-\lambda^2)\cos u \\ &\quad + a_1b_1(1-\lambda^2)\sin^2 u - a_1b_1(1-\lambda^2)\cos^2 u \\ &\quad - \big(a_1^2-b_1^2\big)(1-\lambda^2)\sin u\cos u = 0 \end{aligned} \tag{23}$$

Setting the coefficients of $\cos u$, $\sin u$, $\cos^2 u$, $\sin^2 u$, and $\sin u \cos u$ to be zeros leads to a system of algebraic equations with regard to the undetermined constants a_0, a_1, b_1, and λ as follows

$$\begin{aligned} a_0a_1(1+\lambda^2) &= 0 \\ a_0b_1(1-\lambda^2) &= 0 \\ a_1b_1(1-\lambda^2) &= 0 \\ \big(a_1^2-b_1^2\big)(1-\lambda^2) &= 0 \end{aligned} \tag{24}$$

Solving the aforementioned set of algebraic equations, we obtain the following results

$$a_0 \neq 0, \quad a_1 = 0, \quad b_1 = 0, \quad \lambda \neq 0 \tag{25}$$

$$a_0 = 0, \quad a_1 \neq 0, \quad b_1 \neq 0, \quad \lambda = \pm 1 \tag{26}$$

$$a_1 = 0, \quad a_0 \neq 0, \quad b_1 \neq 0, \quad \lambda = \pm 1 \tag{27}$$

By utilizing Eqs. (7), (10), and (12) combining with Eq. (16), we get the following solutions for the Born–Infeld equation (1)

Family 1:

$$u(x, t) = a_0(x - \lambda t) + \xi_0 \tag{28}$$

where a_0, λ, and ξ_0 are arbitrary constants.

Family 2:

$$u(x, t) = 2 \arctan \left[\pm \sqrt{\frac{a_0 + b_1}{a_0 - b_1}} \tan \frac{\sqrt{a_0^2 - b_1^2}}{2} \eta \right] + \frac{\pi}{2} \qquad \text{for } a_0^2 > b_1^2 \tag{29}$$

$$u(x, t) = 2 \arctan \left[\pm \sqrt{\frac{a_0 + b_1}{a_0 - b_1}} \cot \frac{\sqrt{a_0^2 - b_1^2}}{2} \eta \right] + \frac{\pi}{2} \qquad \text{for } a_0^2 > b_1^2 \tag{30}$$

$$u(x, t) = 2 \arctan \left[\pm \sqrt{\frac{a_0 + b_1}{a_0 - b_1}} \left(\tan \sqrt{a_0^2 - b_1^2} \eta \pm \sec \sqrt{a_0^2 - b_1^2} \eta \right) \right] + \frac{\pi}{2} \quad \text{for } a_0^2 > b_1^2 \tag{31}$$

$$u(x, t) = 2 \arctan \left[\pm \sqrt{\frac{a_0 + b_1}{a_0 - b_1}} \left(\cot \sqrt{a_0^2 - b_1^2} \eta \pm \csc \sqrt{a_0^2 - b_1^2} \eta \right) \right] + \frac{\pi}{2} \quad \text{for } a_0^2 > b_1^2 \tag{32}$$

$$u(x, t) = 2 \arctan \left[\pm \sqrt{\frac{a_0 + b_1}{a_0 - b_1}} \frac{1 \mp \cos \sqrt{a_0^2 - b_1^2} \eta}{\sin \sqrt{a_0^2 - b_1^2} \eta} \right] + \frac{\pi}{2} \quad \text{for } a_0^2 > b_1^2 \tag{33}$$

$$u(x, t) = 2 \arctan \left[\pm \sqrt{\frac{a_0 + b_1}{a_0 - b_1}} \frac{\sin \sqrt{a_0^2 - b_1^2} \eta}{1 \pm \cos \sqrt{a_0^2 - b_1^2} \eta} \right] + \frac{\pi}{2} \quad \text{for } a_0^2 > b_1^2 \tag{34}$$

$$u(x,t) = 2\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\tanh\frac{\sqrt{b_1^2-a_0^2}}{2}\eta\right]+\frac{\pi}{2} \qquad \text{for } a_0^2 < b_1^2 \tag{35}$$

$$u(x,t) = 2\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\coth\frac{\sqrt{b_1^2-a_0^2}}{2}\eta\right]+\frac{\pi}{2} \qquad \text{for } a_0^2 < b_1^2 \tag{36}$$

$$u(x,t) = 2\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\left(\tanh\sqrt{b_1^2-a_0^2}\eta \pm i\sec h\sqrt{b_1^2-a_0^2}\eta\right)\right]+\frac{\pi}{2}$$
$$\text{for } a_0^2 < b_1^2 \tag{37}$$

$$u(x,t) = 2\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\left(\coth\sqrt{b_1^2-a_0^2}\eta \pm \csc h\sqrt{b_1^2-a_0^2}\eta\right)\right]+\frac{\pi}{2}$$
$$\text{for } a_0^2 < b_1^2 \tag{38}$$

$$u(x,t) = 2\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\frac{\cosh\sqrt{b_1^2-a_0^2}\eta \mp 1}{\sinh\sqrt{b_1^2-a_0^2}\eta}\right]+\frac{\pi}{2}$$
$$\text{for } \mathrm{a}_0^2 < b_1^2 \tag{39}$$

$$u(x,t) = 2\arctan\left[\pm\sqrt{\frac{b_1+a_0}{b_1-a_0}}\frac{\sinh\sqrt{b_1^2-a_0^2}\eta}{\cosh\sqrt{b_1^2-a_0^2}\eta \pm 1}\right]+\frac{\pi}{2}$$
$$\text{for } a_0^2 < b_1^2 \tag{40}$$

$$u(\xi) = 2\arctan a_0\eta + \frac{\pi}{2} \qquad \text{for } b_1 = a_0 \tag{41}$$

$$u(\xi) = -2\,\mathrm{arccot}\, a_0\eta + \frac{\pi}{2} \qquad \text{for } b_1 = -a_0 \tag{42}$$

where $\eta = x \mp t + \xi_0$, and a_0, b_1, and ξ_0 are arbitrary constants.

Family 3:

$$u(x,t) = 2\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}\tanh\frac{\sqrt{a_1^2+b_1^2}}{2}\eta + \frac{b_1}{a_1}\right] \tag{43}$$

$$u(x,t) = 2\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}\coth\frac{\sqrt{a_1^2+b_1^2}}{2}\eta + \frac{b_1}{a_1}\right] \tag{44}$$

$$u(x,t) = 2\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}\left(\tanh\sqrt{a_1^2+b_1^2}\eta \pm i\sec h\sqrt{a_1^2+b_1^2}\eta\right)+\frac{b_1}{a_1}\right] \tag{45}$$

$$u(x,t) = 2\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}\left(\coth\sqrt{a_1^2+b_1^2}\eta \pm \csc h\sqrt{a_1^2+b_1^2}\eta\right)+\frac{b_1}{a_1}\right] \tag{46}$$

$$u(x,t) = 2\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}\frac{\cosh\sqrt{a_1^2+b_1^2}\eta \mp 1}{\sinh\sqrt{a_1^2+b_1^2}\eta}+\frac{b_1}{a_1}\right] \tag{47}$$

$$u(x,t) = 2\arctan\left[\frac{\sqrt{a_1^2+b_1^2}}{a_1}\frac{\sinh\sqrt{a_1^2+b_1^2}\eta}{\cosh\sqrt{a_1^2+b_1^2}\eta \pm 1}+\frac{b_1}{a_1}\right] \tag{48}$$

where $\eta = x \mp t + \xi_0$, and a_1, b_1, and ξ_0 are arbitrary constants.

Obviously, the solutions (18), (25), (30), and (31) are quite in agreement with those obtained in reference Sirendaoreji and Jiong (2002). The rest of solutions, however, are the first reported ones to the Born–Infeld equation, which can not be seen in literature to the best of our knowledge.

Finally, it should be pointed out that we have verified these solutions obtained in this paper by putting them back into original equation with the aid of Mathematica.

3. CONCLUSIONS

In conclusion, abundant types of new explicit exact solutions to the Born–Infeld equation are presented in a concise manner. The key idea of our method is to take full advantage of the solutions of an extra ordinary differential equation introduced in this paper. Its advantage is to transform the problem for solving a NPDE into the one for solving an ODE whose solutions can be easily found by means of the method of separation of variables. Therefore, the problem in question is significantly simplified. As far as we know, this is the first time results like these have been reported. How to generalize this approach to solve other NPDEs is worthy of further study.

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